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Existence results for fractional neutral integro-differential systems with nonlocal condition through resolvent operators

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Abstract

The manuscript is primarily concerned with the new existence results for fractional neutral integro-differential equation (FNIDE) with nonlocal conditions (NLCs) in Banach spaces. Based on the Banach contraction principle and Krasnoselskii fixed point theorem (FPT) joined with resolvent operators, we develop the main results. Ultimately, an representation is also offered to demonstrate the accomplished theorem.

1 Introduction

In this manuscript, we are dealing with the existence of mild solutions for FNIDE with NLCs of the form

$$D^q\left[u(t)+\mathscr{G}\left(t,u(t),\int_0^t e_1(t,s,u(s))ds\right)\right]=\mathscr{A}u(t) + \mathscr{F}\left(t,u(t),\int_0^t e_2(t,s,u(s))ds\right)+\mathscr{H}\left(t,u(t),\int_0^t e_3(t,s,u(s))ds\right), \tag{1.1}$$

$$u(0)+g(u)=u_0, \tag{1.2}$$

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where $t \in \mathscr{I} = [0, T]$ denotes an operational interval, D^q represents the Caputo fractional derivative of order $0 < q < 1$, $\mathscr{A}$ means a closed linear unbounded operator in Banach space $\mathbb{X}$ with dense domain $\mathscr{D}(\mathscr{A})$, $u_0 \in \mathbb{X}$ and $\mathscr{G}, \mathscr{F}, \mathscr{H} : \mathscr{I} \times \mathbb{X}^2 \to \mathbb{X}$, $e_i : \Delta \times \mathbb{X} \to \mathbb{X}, i = 1, 2, 3$; $g : \mathcal{C}(\mathscr{I}, \mathbb{X}) \to \mathbb{X}$ are continuous, where $\Delta = \{(t, s) : 0 \le s \le t \le T\}$. For curtness let us take $E_i u(t) = \displaystyle\int_0^t e_i(t, s, u(s))ds$, $i = 1, 2, 3$. Moreover the integral equation

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{\mathscr{A} u(s)}{(t-s)^{1-q}} ds, \quad t \ge 0, \tag{1.3}$$

possess an involved resolvent operator $(\mathcal{S}(t))_{t \ge 0}$ on $\mathbb{X}$. From this concept, we imply the following statement.

Definition 1.1. *[1, Definition 1.1.3] A one parameter family of bounded linear operators $(\mathcal{S}(t))_{t \ge 0}$ on $\mathbb{X}$ denotes a resolvent operator for (1.3) if the subsequent conditions are satisfied:*

(a) $\mathcal{S}(\cdot)\xi \in \mathcal{C}([0, \infty), \mathbb{X})$ *and* $\mathcal{S}(0)\xi = \xi$ *for all* $\xi \in \mathbb{X}$*,*

(b) $\mathcal{S}(t)\mathscr{D}(\mathscr{A}) \subset \mathscr{D}(\mathscr{A})$ *and* $\mathscr{A}\mathcal{S}(t)\xi = \mathcal{S}(t)\mathscr{A}\xi$ *for all* $\xi \in \mathscr{D}(\mathscr{A})$ *(the domain of* $\mathscr{A}$ *) and every* $t \ge 0$*.*

(c) for every $\xi \in \mathscr{D}(\mathscr{A})$ *and* $t \ge 0$*,*

$$\mathcal{S}(t)\xi = \xi + \frac{1}{\Gamma(q)} \int_0^t \frac{\mathscr{A}\mathcal{S}(p)\xi}{(t-p)^{1-q}} dp.$$

Example 1.1. *We consider* $\mathbb{X} = L^2[0, \pi]$ *with the norm* $|\cdot|_{L^2}$ *and define the operator* $\mathscr{A} : \mathscr{D}(\mathscr{A}) \subset \mathbb{X} \to \mathbb{X}$ *by* $Aw = w''$ *such that*

$$\mathscr{D}(\mathscr{A}) = \{w \in \mathbb{X} : \quad w'' \in \mathbb{X},\ w(0) = w(\pi) = 0\}.$$

Further

$$\mathscr{A} w = \sum_{n=1}^{\infty} n^2 \langle w, w_n \rangle w_n, \quad w \in \mathscr{D}(\mathscr{A}),$$

where $w_n(s) = \sqrt{\frac{2}{\pi}} \sin(ns)$, $n = 1, 2, \ldots,$. *denotes the orthogonal set of eigenvectors of* $\mathscr{A}$*. We recall that* $\mathscr{A}$ *means the infinitesimal generator of an analytic semigroup* $(T(t))_{t \ge 0}$ *in* $\mathbb{X}$ *namely*

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} \langle w, w_n \rangle w_n, \quad \textit{for all} \quad w \in \mathbb{X}, \quad \textit{and every} \quad t > 0.$$

From this concept, it makes sense that $(T(t))_{t\geq 0}$ denotes a uniformly bounded compact semigroup, so that $R(\lambda, \mathscr{A}) = (\lambda - \mathscr{A})^{-1}$ represents a compact operator for all $\lambda \in \varrho(\mathscr{A})$.

By [1, Example 2.2.1], we notice that

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{\mathscr{A} u(s)}{(t-s)^{1-q}} ds, \quad t \geq 0,$$

has an linked analytic resolvent operator $(\mathcal{S}(t))_{t\geq 0}$ on $\mathbb{X}$. It is denoted by

$$\mathcal{S}(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda t} (\lambda^q - \mathscr{A})^{-1} d\lambda, & t > 0; \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ is a contour consisting of rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\pi, \frac{\pi}{2})$. Definitely, we can find a constant $\mathbb{S}_{\mathscr{A}}$ in a way that $\|[\mathcal{S}'(t) - \mathcal{S}'(s)]x\| \leq \mathbb{S}_{\mathscr{A}} |t-s| \|x\|_{[\mathscr{D}(\mathscr{A})]}$ for every $t, s \geq 0$, where $\| \cdot \|_{[\mathscr{D}(\mathscr{A})]}$ means the graph norm.

1.1 Fractional Differential Equation

Fractional calculus is an emerging field being more than 300 years old, while the investigation of fractional calculus principally concentrates on the area of pure mathematics [2–5]. We recall that Mandelbrot [6] recognized that there are numerous fractional dimension phenomena existing in nature and technology. In perspective of this case the fractional calculus is deeply connected to different fields of science and engineering [7-12].

1.2 Neutral Integro-differential Equation

NDEs emerge in numerous topics of applied mathematics and hence these equations have gotten much consideration amid the most recent couple of decades [13, 14]. We recommend to the reader [15] and the references therein. NIDEs happen in the research of population dynamics, compartmental systems, viscoelasticity and many other areas of technology.

1.3 Nonlocal Conditions

The study of existence of solutions to evolution equations with a nonlocal condition in Banach space was initiated by Byszewski [16]. In Byszewski and Lakshmikantham [17] and the references therein, it's possible to obtain other facts about the significance of nonlocal initial conditions in uses. There have been numerous papers related to this subject [16–19].

1.4 Existence

We recall that the existence, controllability and other qualitative and quantitative properties of FDEs would be the most developing area of desire as it can be seen from [18–22]. Mainly, there are several papers dealing with the problem of the existence of a mild solution for abstract semilinear fractional differential equations [[13, 18, 20, 22]]. But, as outlined in [13], some theories of mild solution are not reasonable; for more details on this problem, we insist the reader to refer [13]. Besides, Zhou and Jiao in [18] investigated a class of fractional neutral evolution equations with nonlocal conditions by taking into account an integral equation which is given in terms of probability density and semigroup theory, they established existence and uniqueness results. This motivate us to study the existence results of the structure (1.1)-(1.2) with help of resolvent operators in Banach spaces. To the best of our insight the existence results for (1.1)-(1.2) in current paper are contemporary.

Unlike the present results, this manuscript presents some other results, namely, we include the integral term in $\mathscr{G}, \mathscr{F}$ and $\mathscr{H}$ and given a suitable idea of mild solution of the model (1.1)-(1.2) under resolvent operators. After that we discuss the existence of mild solutions for FNIDE with NLCs of the design (1.1)-(1.2) under Banach and Krasnoselskii fixed point theorem, and the results in [19] might be observed as the special circumstances.

This paper is organize as it is given below. In Section 2 some basic definitions and results are specified. In section 3, the existence of mild solutions for the model (1.1)-(1.2) is analyzed under Banach and Krasnoselskii fixed point theorem. In Section 4 a proper case is presented to reveal the effectiveness of the abstract techniques.

2 Preliminaries

Definition 2.1. *The Riemann-Liouville fractional order integral operator of order $\mu > 0$, of function $\nu \in L^1(\mathbb{R}_+)$ is given by*

$$I^{\mu}_{0+}\nu(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\nu(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. *The expression of the Caputo fractional derivative of order $\mu > 0, n-1 < \mu < n$, is*

$${}^C D^{\mu}_{0+}\nu(t) = \frac{1}{\Gamma(n-\mu)}\int_0^t (t-s)^{n-\mu-1}\nu^n(s)ds,$$

where the function $\nu(t)$ have absolutely continuous derivatives up to order $(n-1)$. If $0<\mu<1$, then

$$^{C}D^{\mu}_{0+}\nu(t)=\frac{1}{\Gamma(1-\mu)}\int_0^t \frac{\nu'(s)}{(t-s)^{\mu}}ds,$$

where $\nu'(s)=D\nu(s)=\frac{d}{ds}\nu(s)$ and ν is an abstract function with values in $\mathbb{X}$.

Additional information on fractional derivatives and their properties is often observed in [13, 18–20].

Let $\mathcal{C}(\mathscr{I};\mathbb{X})$ signifies the space of all continuous functions from $\mathscr{I}$ into a Banach space $\mathbb{X}$ with the supnorm indicated by $\|\cdot\|_{\mathcal{C}(\mathscr{I};\mathbb{X})}$. The $[\mathscr{D}(\mathscr{A})]$ stands for the domain of $\mathscr{A}$ endowed with the graph norm $\|x\|_{[\mathscr{D}(\mathscr{A})]}=\|x\|+\|\mathscr{A}x\|$. Furthermore, $B_r(x,\mathbb{X})$ symbolizes the closed ball with center at x and radius r in $\mathbb{X}$.

Below, we suppose that the resolvent operator $(\mathcal{S}(t))_{t\geq 0}$ of (1.3) is analytic and compact; see for instance [1, Chapter 2]. Besides, $\|\mathcal{S}'(t)x\|\leq \varphi_{\mathscr{A}}(t)\|x\|_{[\mathscr{D}(\mathscr{A})]}$ for all $t\geq 0$ and $\|[\mathcal{S}'(t)-\mathcal{S}'(s)]x\|\leq \mathbb{S}_{\mathscr{A}}|t-s|\|x\|_{[\mathscr{D}(\mathscr{A})]}$ for all $(t,s)\geq 0$, where $\varphi_{\mathscr{A}}$ is a function in $L^1_{\text{loc}}([0,\infty);\mathbb{R}^+)$ and $\mathbb{S}_{\mathscr{A}}$ represents a constant.

Let us discuss

$$u(t)=\frac{1}{\Gamma(q)}\int_0^t \frac{\mathscr{A}u(s)}{(t-s)^{1-q}}ds+w(t),\quad t\in\mathscr{I}, \tag{2.1}$$

where $w\in\mathcal{C}(\mathscr{I};\mathbb{X})$. Utilising [1, Definition 1.1.1], we notice the subsequent idea of mild solution.

Definition 2.3. *A function $u\in\mathcal{C}(\mathscr{I};\mathbb{X})$ is called a mild solution of the integral equation (2.1) on $\mathscr{I}$ provided that $\int_0^t(t-s)^{q-1}u(s)ds\in\mathscr{D}(\mathscr{A})$ for all $t\in\mathscr{I}$ and*

$$u(t)=\frac{\mathscr{A}}{\Gamma(q)}\int_0^t \frac{u(s)}{(t-s)^{1-q}}ds+w(t),\quad t\in\mathscr{I}.$$

The following result has vital impact in our improvement, which takes after from [13, Lemma 1.1].

Lemma 1. *Suppose that the resolvent operator $(\mathcal{S}(t))_{t\geq 0}$ of (1.3) is analytic and compact, and $w\in\mathcal{C}(\mathscr{I};\mathscr{D}(\mathscr{A}))$, then the function $u:\mathscr{I}\to\mathbb{X}$ defined by*

$$u(t)=\int_0^t \mathcal{S}'(t-s)w(s)ds+w(t),\quad t\in\mathscr{I},$$

denotes a mild solution of (2.1).

The idea of a mild solution of (1.1)-(1.2) will be introduced. This equation is equivalent to the subsequent integral equation

$$u(t) = u_0 - g(u) + \mathscr{G}(0, u_0, 0) - \mathscr{G}(t, u(t), E_1 u(t)) + \frac{1}{\Gamma(q)} \int_0^t \frac{\mathscr{A} u(s)}{(t-s)^{1-q}} ds$$
$$+ \frac{1}{\Gamma(q)} \int_0^t \frac{\left[\mathscr{F}(s, u(s), E_2 u(s)) + \mathscr{H}(s, u(s), E_3 u(s))\right]}{(t-s)^{1-q}} ds, \quad t \in \mathscr{I}. \tag{2.1}$$

Definition 2.4. *A function $u \in \mathcal{C}(\mathscr{I}, \mathbb{X})$ is said to be a mild solution of (1.1)-(1.2) on $\mathscr{I}$ if $\int_0^t (t-s)^{q-1} u(s) ds \in \mathscr{D}(\mathscr{A})$ for all $t \in \mathscr{I}$ and fulfills the integral equation (2.1).*

To streamline our advancement, in whatever remains of this work, for a function $u \in \mathcal{C}(\mathscr{I}; \mathbb{X})$ we use the notations $G_u, F_u : \mathscr{I} \to \mathbb{X}$ given by

$$G_u(t) = u_0 - g(u) + \mathscr{G}(0, u_0, 0) - \mathscr{G}(t, u(t), E_1 u(t)),$$
$$F_u(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{\left[\mathscr{F}(s, u(s), E_2 u(s)) + \mathscr{H}(s, u(s), E_3 u(s))\right]}{(t-s)^{1-q}} ds.$$

Utilising the Lemma 2.1, one sees that

$$u(t) = G_u(t) + F_u(t) + \int_0^t \mathcal{S}'(t-s)[G_u(s) + F_u(s)] ds, \quad t \in \mathscr{I},$$

is a mild solution of (1.1)-(1.2).

3 The existence results

Below, we exhibit and show the existence of solutions for the structure (1.1)-(1.2) under Banach and Krasnoselskii's fixed point theorem.

The following suppositions are given:

(H1) The functions $\mathscr{G}, \mathscr{F}, \mathscr{H} : \mathscr{I} \times \mathbb{X}^2 \to \mathscr{D}(\mathscr{A})$ are (completely) continuous, there exist constants $\mathscr{L}_{\mathscr{G}}, \widetilde{\mathscr{L}_{\mathscr{G}}}, \mathscr{L}_{\mathscr{F}}, \widetilde{\mathscr{L}_{\mathscr{F}}}, \mathscr{L}_{\mathscr{H}}, \widetilde{\mathscr{L}_{\mathscr{H}}} > 0$ in ways that for all $(t, x_i, y_i) \in \mathscr{I} \times \mathbb{X}^2$, $i = 1, 2$, we sustain

(a) $\|\mathscr{G}(t, x_1, y_1) - \mathscr{G}(t, x_2, y_2)\|_{[\mathscr{D}(\mathscr{A})]} \leq \mathscr{L}_{\mathscr{G}} \|x_1 - x_2\| + \widetilde{\mathscr{L}_{\mathscr{G}}} \|y_1 - y_2\|$;

(b) $\|\mathscr{F}(t, x_1, y_1) - \mathscr{F}(t, x_2, y_2)\|_{[\mathscr{D}(\mathscr{A})]} \leq \mathscr{L}_{\mathscr{F}} \|x_1 - x_2\| + \widetilde{\mathscr{L}_{\mathscr{F}}} \|y_1 - y_2\|$;

(c) $\|\mathscr{H}(t, x_1, y_1) - \mathscr{H}(t, x_2, y_2)\|_{[\mathscr{D}(\mathscr{A})]} \leq \mathscr{L}_{\mathscr{H}} \|x_1 - x_2\| + \widetilde{\mathscr{L}_{\mathscr{H}}} \|y_1 - y_2\|$;

with

$$\mathcal{C}_{\mathscr{G}} = \max_{t\in\mathscr{I}} \mathscr{G}(t,0,0), \quad \mathcal{C}_{\mathscr{F}} = \max_{t\in\mathscr{I}} \mathscr{F}(t,0,0) \quad \text{and} \quad \mathcal{C}_{\mathscr{H}} = \max_{t\in\mathscr{I}} \mathscr{H}(t,0,0).$$

(H2) The functions e_i: $\Delta \times \mathbb{X} \to \mathscr{D}(\mathscr{A})$, $i = 1,2,3$; are continuous and there exist constants $\mathscr{L}_{e_i} > 0$ such that for all $(t,s,x_j) \in \Delta \times \mathbb{X}, j = 1,2$;

$$\left\| \int_0^t \big[e_i(t,s,x_1) - e_i(t,s,x_2)\big] ds \right\|_{[\mathscr{D}(\mathscr{A})]} \leq \mathscr{L}_{e_i} \|x_1 - x_2\|,$$

with $\mathcal{C}_i = \max_{t\in\mathscr{I}} \int_0^t e_i(t,s,0)ds$, $i = 1,2,3$.

(H3) There exists a constant $\mathscr{L}_g > 0$ of the function $g : \mathcal{C}(\mathscr{I}, \mathbb{X}) \to \mathscr{D}(\mathscr{A})$, in a way that

$$\|g(x_1) - g(x_2)\|_{[\mathscr{D}(\mathscr{A})]} \leq \mathscr{L}_g \|x_1 - x_2\|, \quad \text{for all} \quad x_1, x_2 \in \mathbb{X}.$$

(H4) The following inequalities holds:

(i) Let

$$\begin{aligned}(1 + \|\varphi_{\mathscr{A}}\|_{L^1})\Big(&\|u_0\| + \|g(0)\| + \|\mathscr{G}(0,u_0,0)\| + \widetilde{\mathscr{L}_{\mathscr{G}}}\mathcal{C}_1 + \mathcal{C}_{\mathscr{G}} \\ &+ r(\mathscr{L}_g + \mathscr{L}_{\mathscr{G}} + \widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1}) + \Lambda\big[r(\mathscr{L}_{\mathscr{F}} + \mathscr{L}_{\mathscr{H}} + \widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2} + \widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3}) \\ &+ \widetilde{\mathscr{L}_{\mathscr{F}}}\mathcal{C}_2 + \widetilde{\mathscr{L}_{\mathscr{H}}}\mathcal{C}_3 + \mathcal{C}_{\mathscr{F}} + \mathcal{C}_{\mathscr{H}}\big]\Big) \leq r,\end{aligned}$$

where $\Lambda = \frac{T^q}{\Gamma(q+1)}$ and for some $r > 0$.

(ii) Let

$$\begin{aligned}\Omega = (1 + \|\varphi_{\mathscr{A}}\|_{L^1})\Big(&\mathscr{L}_g + \mathscr{L}_{\mathscr{G}} + \widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1} + \Lambda\big[\mathscr{L}_{\mathscr{F}} + \mathscr{L}_{\mathscr{H}} + \widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2} \\ &+ \widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3}\big]\Big) < 1\end{aligned}$$

be such that $0 \leq \Omega < 1$.

First, we present and prove the uniqueness result.

Theorem 3.1. *Assume that the conditions (H1)-(H4) hold and $u_0 \in \mathscr{D}(\mathscr{A})$. Then the problem (1.1)-(1.2) has a unique mild solution on $\mathscr{I}$.*

Proof. Recognize the operator $\Upsilon : \mathcal{C}(\mathscr{I}, \mathbb{X}) \to \mathcal{C}(\mathscr{I}, \mathbb{X})$ by

$$\Upsilon u(t) = G_u(t) + F_u(t) + \int_0^t \mathcal{S}'(t-s)[G_u(s) + F_u(s)]ds, \quad t \in \mathscr{I}.$$

In perspective of Lemma 2.1 and the argument above, it is easy to see that the operator Υ having a fixed point. Let $Z = \mathcal{C}(\mathscr{I}, \mathbb{X})$ and $B_r(0, Z) = \{z \in Z : \|z\| \leq r\}$. Initially, we demonstrate that Υ maps $B_r(0, Z)$ into $B_r(0, Z)$. For any $u \in Z$, we have

$$\begin{aligned}\|(\Upsilon u)(t)\| &\leq \|G_u(t)\| + \|F_u(t)\| + \left\|\int_0^t \mathcal{S}'(t-s)[G_u(s) + F_u(s)]ds\right\| \\ &\leq I_1 + I_2 + I_3. \end{aligned} \tag{3.1}$$

Now, we calculate the estimations:

$$\begin{aligned} I_1 = \|G_u(t)\| &\leq \|u_0\| + \|g(u)\| + \|\mathscr{G}(0, u_0, 0)\| + \|\mathscr{G}(t, u(t), E_1 u(t))\| \\ &\leq \|u_0\| + \|g(0)\| + \|\mathscr{G}(0, u_0, 0)\| + \widetilde{\mathscr{L}_{\mathscr{G}}}\mathcal{C}_1 + \mathcal{C}_{\mathscr{G}} \\ &\quad + r(\mathscr{L}_g + \mathscr{L}_{\mathscr{G}} + \widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1}), \end{aligned}$$

since

$$\begin{aligned} &\|\mathscr{G}(t, u(t), E_1 u(t))\| \\ &\leq \left\|\mathscr{G}\left(t, u(t), \int_0^t e_1(t, s, u(s))ds\right) - \mathscr{G}(t, 0, 0)\right\| + \|\mathscr{G}(t, 0, 0)\| \\ &\leq \mathscr{L}_{\mathscr{G}} r + \widetilde{\mathscr{L}_{\mathscr{G}}}\left\| \int_0^t e_1(t, s, u(s))ds\right\| + \mathcal{C}_{\mathscr{G}} \\ &\leq \mathscr{L}_{\mathscr{G}} r + \widetilde{\mathscr{L}_{\mathscr{G}}}\left[\left\| \int_0^t \left[e_1(t, s, u(s)) - e_1(t, s, 0)\right]ds\right\| + \left\| \int_0^t e_1(t, s, 0)ds\right\|\right] \\ &\quad + \mathcal{C}_{\mathscr{G}} \\ &\leq \mathscr{L}_{\mathscr{G}} r + \widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1} r + \widetilde{\mathscr{L}_{\mathscr{G}}}\mathcal{C}_1 + \mathcal{C}_{\mathscr{G}}. \end{aligned}$$

$$\begin{aligned} I_2 = \|F_u(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[\|\mathscr{F}(s, u(s), E_2 u(s))\| \\ &\quad + \|\mathscr{H}(s, u(s), E_3 u(s))\|\big]ds \\ &\leq \Lambda\big[r(\mathscr{L}_{\mathscr{F}} + \mathscr{L}_{\mathscr{H}} + \widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2} + \widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3}) + \widetilde{\mathscr{L}_{\mathscr{F}}}\mathcal{C}_2 + \widetilde{\mathscr{L}_{\mathscr{H}}}\mathcal{C}_3 \\ &\quad + \mathcal{C}_{\mathscr{F}} + \mathcal{C}_{\mathscr{H}}\big], \end{aligned}$$

since

$$\begin{aligned}
&\frac{1}{\Gamma(q)}\int_0^t \frac{\|\mathscr{F}(s,u(s),E_2u(s))\|}{(t-s)^{1-q}}ds\\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[\|\mathscr{F}(s,u(s),E_2u(s))-\mathscr{F}(s,0,0)\|+\|\mathscr{F}(s,0,0)\|\big]ds\\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[\mathscr{L}_{\mathscr{F}}\|u(s)\|+\widetilde{\mathscr{L}_{\mathscr{F}}}\|E_2u(s)\|+\mathcal{C}_{\mathscr{F}}\big]ds\\
&\le \frac{T^q}{\Gamma(q+1)}\left[\mathscr{L}_{\mathscr{F}}r+\widetilde{\mathscr{L}_{\mathscr{F}}}\left\|\int_0^t e_2(t,s,u(s))ds\right\|+\mathcal{C}_{\mathscr{F}}\right]\\
&\le \Lambda\Bigg[\mathscr{L}_{\mathscr{F}}r+\widetilde{\mathscr{L}_{\mathscr{F}}}\left[\left\|\int_0^t \big[e_2(t,s,u(s))-e_2(t,s,0)\big]ds\right\|+\left\|\int_0^t e_2(t,s,0)ds\right\|\right]\\
&\quad+\mathcal{C}_{\mathscr{F}}\Bigg]\\
&\le \Lambda\Big[\mathscr{L}_{\mathscr{F}}r+\widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2}r+\widetilde{\mathscr{L}_{\mathscr{F}}}\mathcal{C}_2+\mathcal{C}_{\mathscr{F}}\Big].
\end{aligned}$$

In the same way, we receive

$$\begin{aligned}
&\frac{1}{\Gamma(q)}\int_0^t \frac{\|\mathscr{H}(s,u(s),E_3u(s))\|}{(t-s)^{1-q}}ds\\
&\le \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\big[\|\mathscr{H}(s,u(s),E_3u(s))-\mathscr{H}(s,0,0)\|+\|\mathscr{H}(s,0,0)\|\big]ds\\
&\le \Lambda\Big[\mathscr{L}_{\mathscr{H}}r+\widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3}r+\widetilde{\mathscr{L}_{\mathscr{H}}}\mathcal{C}_3+\mathcal{C}_{\mathscr{H}}\Big].
\end{aligned}$$

Furthermore, we obtain

$$\begin{aligned}
I_3 &= \left\|\int_0^t \mathcal{S}'(t-s)[G_u(s)+F_u(s)]ds\right\|\\
&\le \|\varphi_{\mathscr{A}}\|_{L^1}\Big(\|u_0\|+\|g(0)\|+\|\mathscr{G}(0,u_0,0)\|+\widetilde{\mathscr{L}_{\mathscr{G}}}\mathcal{C}_1+\mathcal{C}_{\mathscr{G}}\\
&\quad+r(\mathscr{L}_g+\mathscr{L}_{\mathscr{G}}+\widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1})+\Lambda\big[r(\mathscr{L}_{\mathscr{F}}+\mathscr{L}_{\mathscr{H}}+\widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2}+\widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3})\\
&\quad+\widetilde{\mathscr{L}_{\mathscr{F}}}\mathcal{C}_2+\widetilde{\mathscr{L}_{\mathscr{H}}}\mathcal{C}_3+\mathcal{C}_{\mathscr{F}}+\mathcal{C}_{\mathscr{H}}\big]\Big).
\end{aligned}$$

Therefore equation (3.1) becomes

$$\begin{aligned}
\|(\Upsilon u)(t)\| \leq (1+\|\varphi_{\mathscr{A}}\|_{L^1})\bigg(&\|u_0\|+\|g(0)\|+\|\mathscr{G}(0,u_0,0)\|+\widetilde{\mathscr{L}_{\mathscr{G}}}\mathcal{C}_1+\mathcal{C}_{\mathscr{G}}\\
&+r(\mathscr{L}_g+\mathscr{L}_{\mathscr{G}}+\widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1})+\Lambda\big[r(\mathscr{L}_{\mathscr{F}}+\mathscr{L}_{\mathscr{H}}+\widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2}+\widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3})\\
&+\widetilde{\mathscr{L}_{\mathscr{F}}}\mathcal{C}_2+\widetilde{\mathscr{L}_{\mathscr{H}}}\mathcal{C}_3+\mathcal{C}_{\mathscr{F}}+\mathcal{C}_{\mathscr{H}}\big]\bigg)\\
\leq r.
\end{aligned}$$

Thus Υ maps $B_r(0,Z)$ into $B_r(0,Z)$ and so $\Upsilon u \in B_r$. From the estimation I_3, we see that the function $s \to \mathcal{S}'(t-s)(G_u(s)+F_u(s))$ is integrable on $[0,t]$ for all $t \in \mathscr{I}$ and $\Upsilon u \in Z$. Since $F_u(\cdot)$ and $G_u(\cdot)$ are continuous and hence Υ is well defined. Finally, we show that Υ is a contraction on $B_r(0,Z)$. For this, let us consider $u,v \in Z$ and $t \in \mathscr{I}$, we sustain

$$\begin{aligned}
&\|(\Upsilon u)(t)-(\Upsilon v)(t)\|\\
&\leq \|G_u(t)-G_v(t)\|+\|F_u(t)-F_v(t)\|+\int_0^t \|\mathcal{S}'(t-s)[G_u(s)-G_v(s)]\|\,ds\\
&\quad+\int_0^t \|\mathcal{S}'(t-s)[F_u(s)-F_v(s)]\|\,ds\\
&\quad\leq (1+\|\varphi_{\mathscr{A}}\|_{L^1})\bigg(\mathscr{L}_g+\mathscr{L}_{\mathscr{G}}+\widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1}+\Lambda\big[\mathscr{L}_{\mathscr{F}}+\mathscr{L}_{\mathscr{H}}+\widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2}\\
&\qquad+\widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3}\big]\bigg)\|u-v\|\\
&\quad\leq \Omega\|u-v\|.
\end{aligned}$$

From the assumption (H4) and in the perspective of the contraction mapping principle, we conclude that Υ includes a unique fixed point $u \in Z$ which represents a mild solution (1.1)-(1.2) on $\mathscr{I}$. □

Next, we present some generally existence results. As a result, we utilize a fixed point theorem due to Krasnoselskii [18].

Theorem 3.2. *Suppose that the conditions (H1)-(H4) hold and $u_0 \in \mathscr{D}(\mathscr{A})$. Then the problem (1.1)-(1.2) has at least a mild solution on $\mathscr{I}$ provided that*

$$(1+\|\varphi_{\mathscr{A}}\|_{L^1})\Big[\mathscr{L}_g+\mathscr{L}_{\mathscr{G}}+\widetilde{\mathscr{L}_{\mathscr{G}}}\mathscr{L}_{e_1}\Big]<1. \tag{3.2}$$

Proof. Now, we define the operator $\overline{\Upsilon}$ is same as defined in Theorem 3.1. In order to apply Krasnoselskii theorem, we need to split the operator $\overline{\Upsilon}$ as $\overline{\Upsilon}_1 + \overline{\Upsilon}_2$ on $B_r(0, Z)$, where

$$\overline{\Upsilon}_1 u(t) = G_u(t) + \int_0^t \mathcal{S}'(t-s) G_u(s) ds$$

and

$$\overline{\Upsilon}_2 u(t) = F_u(t) + \int_0^t \mathcal{S}'(t-s) F_u(s) ds.$$

From first part of Theorem 3.1, we note that $\overline{\Upsilon}$ maps $B_r(0, Z)$ into $B_r(0, Z)$ and for any $u, v \in Z$, we have $\overline{\Upsilon}_1 u + \overline{\Upsilon}_2 v$ is also belongs to B_r, since

$$\begin{aligned}
&\|\overline{\Upsilon}_1 u(t) + \overline{\Upsilon}_2 v(t)\| \\
&\leq (1 + \|\varphi_{\mathscr{A}}\|_{L^1}) \Big(\|u_0\| + \|g(0)\| + \|\mathscr{G}(0, u_0, 0)\| + \widetilde{\mathscr{L}_{\mathscr{G}}} \mathcal{C}_1 + \mathcal{C}_{\mathscr{G}} \\
&\quad + r(\mathscr{L}_g + \mathscr{L}_{\mathscr{G}} + \widetilde{\mathscr{L}_{\mathscr{G}}} \mathscr{L}_{e_1}) + \Lambda \big[r(\mathscr{L}_{\mathscr{F}} + \mathscr{L}_{\mathscr{H}} + \widetilde{\mathscr{L}_{\mathscr{F}}} \mathscr{L}_{e_2} + \widetilde{\mathscr{L}_{\mathscr{H}}} \mathscr{L}_{e_3}) \\
&\quad + \widetilde{\mathscr{L}_{\mathscr{F}}} \mathcal{C}_2 + \widetilde{\mathscr{L}_{\mathscr{H}}} \mathcal{C}_3 + \mathcal{C}_{\mathscr{F}} + \mathcal{C}_{\mathscr{H}} \big] \Big) \\
&\leq r.
\end{aligned}$$

From the estimation I_3 of Theorem 3.1, we notice that

$$\begin{aligned}
&\left\| \int_0^t \mathcal{S}'(t-s) G_u(s) ds \right\| \\
&\leq \|\varphi_{\mathscr{A}}\|_{L^1} \Big[\|u_0\| + \|g(0)\| + \|\mathscr{G}(0, u_0, 0)\| + \widetilde{\mathscr{L}_{\mathscr{G}}} \mathcal{C}_1 + \mathcal{C}_{\mathscr{G}} \\
&\quad + r(\mathscr{L}_g + \mathscr{L}_{\mathscr{G}} + \widetilde{\mathscr{L}_{\mathscr{G}}} \mathscr{L}_{e_1}) \Big]
\end{aligned}$$

which suggests that the function $s \to \mathcal{S}'(t-s) G_u(s)$ is integrable on $[0, t]$ for all $t \in \mathscr{I}$ and $\overline{\Upsilon}_1 u \in Z$.

Furthermore, for any $u, v \in Z$ and from second part of Theorem 3.1, we obtain

$$\|\overline{\Upsilon}_1 u(t) - \overline{\Upsilon}_1 v(t)\| \leq (1 + \|\varphi_{\mathscr{A}}\|_{L^1}) \Big[\mathscr{L}_g + \mathscr{L}_{\mathscr{G}} + \widetilde{\mathscr{L}_{\mathscr{G}}} \mathscr{L}_{e_1} \Big] \|u - v\|.$$

From (3.1), we observe that $\overline{\Upsilon}_1$ is a contraction on $B_r(0, Z)$.

Next, we demonstrate that the operator $\overline{\Upsilon}_2$ is completely continuous. From I_3 of Theorem 3.1, we realize that the function $s \to \int_0^t \mathcal{S}'(t-s)F_u(s)ds$ is integrable on $[0,t]$ for all $t \in \mathscr{I}$. Initially, we show that $\overline{\Upsilon}_2$ is uniformly bounded. From I_2 of Theorem 3.1, for $t \in \mathscr{I}$, we sustain

$$\begin{aligned}\|\overline{\Upsilon}_2 u(t)\| &\leq \|F_u(t)\| + \left\|\int_0^t \mathcal{S}'(t-s)F_u(s)ds\right\| \\ &\leq (1+\|\varphi_{\mathscr{A}}\|_{L^1})\Lambda\big[r(\mathscr{L}_{\mathscr{F}} + \mathscr{L}_{\mathscr{H}} + \widetilde{\mathscr{L}_{\mathscr{F}}}\mathscr{L}_{e_2} \\ &\quad + \widetilde{\mathscr{L}_{\mathscr{H}}}\mathscr{L}_{e_3}) + \widetilde{\mathscr{L}_{\mathscr{F}}}\mathcal{C}_2 + \widetilde{\mathscr{L}_{\mathscr{H}}}\mathcal{C}_3 + \mathcal{C}_{\mathscr{F}} + \mathcal{C}_{\mathscr{H}}\big].\end{aligned}$$

This shows that $\overline{\Upsilon}_2$ is uniformly bounded.

Now, we prove that the operator $\overline{\Upsilon}_2$ is continuous. To prove this, let $\{u_n\}$ be a sequence in $B_r(0,Z)$ such that $u_n \to u$ in $B_r(0,Z)$. Since the functions $\mathscr{F}, \mathscr{H}, e_2$, and e_3 are continuous, $\mathscr{F}(s,u_n(s),E_2u_n(s)) \to \mathscr{F}(s,u(s),E_2u(s))$ and $\mathscr{H}(s,u_n(s),E_3u_n(s)) \to \mathscr{H}(s,u(s),E_3u(s))$ as $n \to \infty$. Now for each $t \in \mathscr{I}$, we have

$$\begin{aligned}&\|\overline{\Upsilon}_2 u_n(t) - \overline{\Upsilon}_2 u(t)\| \\ &\leq \frac{1}{\Gamma(q)}\int_0^t \frac{\|\mathscr{F}(s,u_n(s),E_2u_n(s)) - \mathscr{F}(s,u(s),E_2u(s))\|}{(t-s)^{1-q}}ds\end{aligned}$$

$$\begin{aligned}&+ \frac{1}{\Gamma(q)}\int_0^t \frac{\|\mathscr{H}(s,u_n(s),E_3u_n(s)) - \mathscr{H}(s,u(s),E_3u(s))\|}{(t-s)^{1-q}}ds \\ &+ \int_0^t \mathcal{S}'(t-s)\bigg[\frac{1}{\Gamma(q)}\int_0^s \frac{\|\mathscr{F}(\tau,u_n(\tau),E_2u_n(\tau)) - \mathscr{F}(\tau,u(\tau),E_2u(\tau))\|}{(s-\tau)^{1-q}}d\tau \\ &+ \frac{1}{\Gamma(q)}\int_0^s \frac{\|\mathscr{H}(\tau,u_n(\tau),E_3u_n(\tau)) - \mathscr{H}(\tau,u(\tau),E_3u(\tau))\|}{(s-\tau)^{1-q}}d\tau\bigg]ds \\ &\to 0 \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence, it is easy to see that $\overline{\Upsilon}_2$ is continuous.

Presently, we want to show that the set $\{\overline{\Upsilon}_2 u(t) : u \in B_r(0,Z)\}$ is relatively compact in $\mathbb{X}$ for all $t \in \mathscr{I}$. Obviously $\{\overline{\Upsilon}_2 u(0) : u \in B_r(0,Z)\}$ is compact.

Fix $t \in (0,T]$ and $u \in B_r(0,Z)$, we recognize the operator $\overline{\Upsilon}_2^\epsilon$ by

$$(\overline{\Upsilon}_2^\epsilon u)(t) = \frac{1}{\Gamma(q)}\int_0^{t-\epsilon}\frac{\mathscr{F}(s,u(s),E_2u(s))}{(t-s)^{1-q}}ds + \frac{1}{\Gamma(q)}\int_0^{t-\epsilon}\frac{\mathscr{H}(s,u(s),E_3u(s))}{(t-s)^{1-q}}ds$$
$$+ \int_0^{t-\epsilon}\mathcal{S}'(t-s)\bigg[\frac{1}{\Gamma(q)}\int_0^s\frac{\mathscr{F}(\tau,u(\tau),E_2u(\tau))}{(s-\tau)^{1-q}}d\tau$$
$$+ \frac{1}{\Gamma(q)}\int_0^s\frac{\mathscr{H}(\tau,u(\tau),E_3u(\tau))}{(s-\tau)^{1-q}}d\tau\bigg]ds.$$

From $(H1)(b)(c)$, we note that the functions $\mathscr{F}(\cdot),\mathscr{H}(\cdot)$ are completely continuous, the set $V_\epsilon = \{\overline{\Upsilon}_2^\epsilon u(t) : u \in B_r(0,Z)\}$ is precompact in $\mathbb{X}$, for any $\epsilon > 0$, $0 < \epsilon < t$. In addition, for every $u(\cdot) \in B_r(0,Z)$, we receive

$$\|\overline{\Upsilon}_2 u(t) - \overline{\Upsilon}_2^\epsilon u(t)\| \leq \frac{1}{\Gamma(q)}\int_{t-\epsilon}^t\frac{\|\mathscr{F}(s,u(s),E_2u(s))\|}{(t-s)^{1-q}}ds$$
$$+ \frac{1}{\Gamma(q)}\int_{t-\epsilon}^t\frac{\|\mathscr{H}(s,u(s),E_3u(s))\|}{(t-s)^{1-q}}ds$$
$$+ \int_{t-\epsilon}^t\|\mathcal{S}'(t-s)\|\bigg[\frac{1}{\Gamma(q)}\int_0^s\frac{\|\mathscr{F}(\tau,u(\tau),E_2u(\tau))\|}{(s-\tau)^{1-q}}d\tau$$
$$+ \frac{1}{\Gamma(q)}\int_0^s\frac{\|\mathscr{H}(\tau,u(\tau),E_3u(\tau))\|}{(s-\tau)^{1-q}}d\tau\bigg]ds.$$

This demonstrates precompact sets V_ϵ are arbitrarily close to the set $\{\overline{\Upsilon}_2 u(t) : u \in B_r(0,Z)\}$. Hence the set $\{\overline{\Upsilon}_2 u(t) : u \in B_r(0,Z)\}$ is precompact in $\mathbb{X}$.

Finally, we show that $\Phi_2(B_r(0,Z))$ is equicontinuous. For our convenience, set

$$\widetilde{\mathscr{F}}(s) = \mathscr{F}(s,u(s),E_2u(s)) \quad \text{and} \quad \widetilde{\mathscr{H}}(s) = \mathscr{H}(s,u(s),E_3u(s)).$$

The functions $\overline{\Upsilon}_2 u, u \in B_r(0,Z)$ are equicontinuous at $t = 0$. For $t < t + h \leq$

$T, h > 0$ we have

$$
\begin{aligned}
&\|\overline{\Upsilon}_2 u(t+h) - \overline{\Upsilon}_2 u(t)\| \\
&\leq \frac{1}{\Gamma(q)} \bigg\| \int_0^{t+h} \frac{\widetilde{\mathscr{F}}(s)}{(t+h-s)^{1-q}} ds + \int_0^{t+h} \frac{\widetilde{\mathscr{H}}(s)}{(t+h-s)^{1-q}} ds \\
&\quad - \int_0^t \frac{\widetilde{\mathscr{F}}(s)}{(t-s)^{1-q}} ds - \int_0^t \frac{\widetilde{\mathscr{H}}(s)}{(t-s)^{1-q}} ds \bigg\| \\
&\quad + \frac{1}{\Gamma(q)} \bigg\| \int_0^{t+h} \mathcal{S}'(t+h-s) \int_0^s \frac{\widetilde{\mathscr{F}}(\tau) + \widetilde{\mathscr{H}}(\tau)}{(t+h-\tau)^{1-q}} d\tau ds \\
&\quad - \int_0^t \mathcal{S}'(t-s) \int_0^s \frac{\widetilde{\mathscr{F}}(\tau) + \widetilde{\mathscr{H}}(\tau)}{(t-\tau)^{1-q}} d\tau ds \bigg\| \\
&\leq \frac{1}{\Gamma(q)} \int_0^t \left[\frac{1}{(t-s)^{1-q}} - \frac{1}{(t+h-s)^{1-q}} \right] \|\widetilde{\mathscr{F}}(s) + \widetilde{\mathscr{H}}(s)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_t^{t+h} \frac{\|\widetilde{\mathscr{F}}(s) + \widetilde{\mathscr{H}}(s)\|}{(t+h-s)^{1-q}} ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^h \|\mathcal{S}'(t+h-s)\| \int_0^s \frac{\|\widetilde{\mathscr{F}}(\tau) + \widetilde{\mathscr{H}}(\tau)\|}{(s-\tau)^{1-q}} d\tau ds \\
&\quad + \frac{1}{\Gamma(q)} \int_0^t \|\mathcal{S}'(t-s)\| \bigg\| \int_0^{s+h} \frac{\widetilde{\mathscr{F}}(\tau)}{(s+h-\tau)^{1-q}} d\tau + \int_0^{s+h} \frac{\widetilde{\mathscr{H}}(\tau)}{(s+h-\tau)^{1-q}} d\tau \\
&\quad - \int_0^s \frac{\widetilde{\mathscr{F}}(\tau)}{(s-\tau)^{1-q}} d\tau - \int_0^s \frac{\widetilde{\mathscr{H}}(\tau)}{(s-\tau)^{1-q}} d\tau \bigg\| ds
\end{aligned}
$$

which tends to zero as $h \to 0$. From (H1)(b)(c), we observe that the functions $\mathscr{F}(\cdot)$ and $\mathscr{H}(\cdot)$ are completely continuous, the set $\{\overline{\Upsilon}_2 u : u \in B_r(0, Z)\}$ is equicontinuous. Hence, we have demonstrated that $\overline{\Upsilon}_2(B_r(0, Z))$ is relatively compact for $t \in \mathscr{I}$. By Arzela-Ascoli's theorem $\overline{\Upsilon}_2$ is compact. Hence by the Krasnoselskii fixed point theorem [18] we can find a fixed point $u \in Z$ in ways that $\overline{\Upsilon} u = u$ which is the mild solution to the structure $(1.1) - (1.2)$. $\square$

4 Example

We discuss the following FNIDE with NLCs of the model

$$\frac{\partial^q}{\partial t^q}\left[u(t,x)+a_1(t)\sin u(t,x)+\int_0^t a_2(t-s)e^{-u(s,x)}ds\right]=\frac{\partial^2}{\partial x^2}u(t,x)$$
$$+a_3(t)\sin u(t,x)+\int_0^t a_4(t-s)e^{-u(s,x)}ds+a_5(t)\sin u(t,x)$$
$$+\int_0^t a_6(t-s)e^{-u(s,x)}ds, \quad t>0 \tag{4.1}$$
$$u(t,0)=u(t,\pi)=0, \quad (t,x)\in[0,T]\times[0,\pi], \tag{4.2}$$
$$u(0,x)+\sum_{i=1}^{n}\int_0^{t_i} b_i(\tau)u(\tau,x)d\tau=z(x), \tag{4.3}$$

where $q\in(0,1),\ z\in L^2[0,\pi],\ a_i,b_i\in L^2(\mathscr{I})$.

In perspective of Example 1.1, it is easy to see that $\mathcal{S}(t)$ is differentiable and there exists a constant $\widetilde{M}>0$ in ways that $\|\mathcal{S}'(t)x\|\leq\widetilde{M}\|x\|$, for $x\in\mathscr{D}(\mathscr{A})$. To represent the differential systems (4.1)-(4.3) in the abstract form (1.1)-(1.2), set $\mathscr{G},\mathscr{F},\mathscr{H}:\mathscr{I}\times\mathbb{X}^2\to\mathbb{X}, g:Z\to\mathbb{X}$ and $e_i:\Delta\times\mathbb{X}\to\mathbb{X}$, $i=1,2,3$; defined by

$$\begin{aligned}
\mathscr{G}(t,w,E_1w)(x)&=w(x)+a_1(t)\sin w(x)+E_1w(x),\\
E_1w(x)&=e_1(t,s,w(x))=a_2(t-s)e^{-w(x)};\\
\mathscr{F}(t,w,E_2w)(x)&=w(x)+a_3(t)\sin w(x)+E_2w(x),\\
E_2w(x)&=e_2(t,s,w(x))=a_4(t-s)e^{-w(x)};\\
\mathscr{H}(t,w,E_3w)(x)&=w(x)+a_5(t)\sin w(x)+E_3w(x),\\
E_3w(x)&=e_3(t,s,w(x))=a_6(t-s)e^{-w(x)}
\end{aligned}$$

and

$$g(w(x))=\sum_{i=1}^{n}\int_0^{t_i} b_i(\tau)u(\tau,x)d\tau.$$

We notice that $\|g(u(x))-g(v(x))\|\leq\sum_{i=1}^{n}t_i\|b_i\|\|u-v\|$. Since $\|\varphi_{\mathscr{A}}\|_{L^1}\leq\widetilde{M}$, $\mathscr{L}_1=(1+\sup_{t\in\mathscr{I}}\|a_1(t)\|+\mathscr{L}_{e_1})$, $\mathscr{L}_1=\max_{t\in\mathscr{I}}\{\mathscr{L}_{\mathscr{G}},\widetilde{\mathscr{L}_{\mathscr{G}}}\}$, $\mathscr{L}_{e_1}=\sup_{t\in\mathscr{I}}\|a_2(t)\|$, $\mathscr{L}_2=(1+\sup_{t\in\mathscr{I}}\|a_3(t)\|+\mathscr{L}_{e_2})$, $\mathscr{L}_2=\max_{t\in\mathscr{I}}\{\mathscr{L}_{\mathscr{F}},\widetilde{\mathscr{L}_{\mathscr{F}}}\}$, $\mathscr{L}_{e_2}=\sup_{t\in\mathscr{I}}\|a_4(t)\|$, $\mathscr{L}_3=$

$(1 + \sup_{t\in\mathscr{I}} \|a_5(t)\| + \mathscr{L}_{e_3})$, $\mathscr{L}_3 = \max_{t\in\mathscr{I}}\{\mathscr{L}_{\mathscr{H}}, \widetilde{\mathscr{L}_{\mathscr{H}}}\}, \mathscr{L}_{e_3} = \sup_{t\in\mathscr{I}} \|a_6(t)\|$ and $\mathscr{L}_g = \sum_{i=1}^{n} t_i\|b_i\|$ and choose t_i in way that

$$\Omega = (1 + \widetilde{M})\bigg(\mathscr{L}_g + \mathscr{L}_1(1 + \mathscr{L}_{e_1}) + \Lambda\big[\mathscr{L}_2(1 + \mathscr{L}_{e_2}) + \mathscr{L}_3(1 + \mathscr{L}_{e_3})\big]\bigg) < 1.$$

Therefore the conditions (H1)-(H4) of Theorem 3.1 are fulfilled. Thus there exists a function $u \in C(\mathscr{I}, L^2[0,\pi])$ which is a mild solution of (4.1)-(4.3) on $\mathscr{I}$.

5 Conclusion

In this manuscript, abstract results concerning the existence results of FNIDE with NLCs of order $0 < q < 1$ in Banach space are obtained. A new set of sufficient conditions for the existence results of the system (1.1)-(1.2) has been formulated and proved by using the fractional calculus, resolvent operators, contraction mapping principle, Krasnoselskii fixed point theorem and semigroup techniques. Finally an application is provided to illustrate the obtained theoretical results. For the future research, it is interesting to investigate the existence and controllability results of FNIDE with nonlocal and impulsive conditions of Sobolev type and time varying delays.

References

[1] J. Pruss, *Evolutionary Integral Equations and Applications: Monographs in Mathematics*, Vol. 87. Birkhauser, Basel, 1993.

[2] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[3] D. Baleanu, J. A. T. Machado and A. C. J. Luo, *Fractional Dynamics and Control*, Springer, New York, USA, 2012.

[4] A.A. Kilbas, H.M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amesterdam, 2006.

[5] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[6] B.B. Mandelbrot, *The Fractal Geometry of Nature*, First Edition., W. H. Freeman & Co. Ltd., New York, 1974.

[7] S.K. Agrawal, H.M. Srivastava and S. Das, Synchronization of fractional order chaotic systems using active control method, *Chaos Solitons Fractals*, 45(2012) 737–752.

[8] D. Guyomar, B. Ducharne, G. Sebald and D. Audiger, Fractional derivative operators for modeling the dynamic polarization behavior as a function of frequency and electric field amplitude, *IEEE Trans. Ultrason. Ferroelectr. Freq. Control*, 56(2009), 437–443.

[9] I.S. Jesus and J.A.T. Machado, Implementation of fractional-order electromagnetic potential through a genetic algorithm, *Commun. Nonlinear Sci. Numer. Simul.,* 14(2009), 1838–1843.

[10] F.C. Meral, T.J. Royston and R. Magin, Fractional calculus in viscoelasticity: An experimental study, *Commun. Nonlinear Sci. Numer. Simul.,* 15(2010), 939–945.

[11] M.D. Ortigueira, C.M. Ionescu, J.T. Machado and J.J. Trujillo, Fractional signal processing and applications, *Signal Process*, 107(2015) 197–197.

[12] F. Wang, D. Chen, X. Zhang and Y. Wub, The existence and uniqueness theorem of the solution to a class of nonlinear fractional order system with time delay, *Applied Mathematics Letters*, 53(2016), 45-51.

[13] E. Hernandez, D.O' Regan and K. Balachandran, On recent developments in the theory of abstract differential equations with fractional derivatives, *Nonlinear Anal.,* 73(2010), 3462–3471.

[14] E. Hernandez and H.R. Henriquez, Impulsive partial neutral differential equations, *Applied Mathematical Letters*, 19(2006), 215–222.

[15] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional-Differential Equations*, Springer-Verlag, New York, 1993.

[16] L. Byszewski, Theorems about existence and uniqueness of solutions of a semi-linear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.,* 162(1991), 494–505.

[17] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Appl. Anal.,* 40(1991), 11–19.

[18] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.*, 59(3)(2010), 1063–1077.

[19] K. Balachandran and S. Kiruthika, Existence results for fractional integrodifferential equations with nonlocal condition via resolvent operators, *Computers and Mathematics with Applications*, 62(2011), 1350–1358.

[20] S. Suganya, D. Baleanu and M. Mallika Arjunan, A note on fractional neutral integro-differential inclusions with state-dependent delay in Banach spaces, *Journal of Computational Analysis & Applications*, 20(1)(2016), 1302–1317.

[21] P. Balasubramaniam and P. Tamilalagan, Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi function, *Applied Mathematics and Computation*, 256(2015), 232–2466.

[22] X.B. Shu and F. Xu, The existence of solutions for impulsive fractional partial neutral differential equations, *Journal of Mathematics*, Volume 2013, Article ID 147193, 9 pages.

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